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Author(s): Rafael del Río Castillo

Source: *American Journal of Mathematics*, Vol. 113, No. 2 (Apr., 1991), pp. 203–217

Published by: [The Johns Hopkins University Press](#)

Stable URL: <http://www.jstor.org/stable/2374905>

Accessed: 28/09/2010 18:42

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SINGULAR SPECTRUM OF STURM-LIOUVILLE OPERATORS UNDER LOCAL PERTURBATIONS

By RAFAEL DEL RÍO CASTILLO

1. Introduction. Let us consider the selfadjoint realizations of the differential expression

$$(lu)(x) = -u''(x) + q(x)u(x) \quad x \in [0, \infty)$$

where q is a real valued, locally integrable function defined in $[0, \infty)$. The endpoint 0 is regular and we assume that the limit point case occurs at ∞ .

It was shown in [1] and [5] that if we vary the boundary condition at zero, that is, if we vary the selfadjoint realization of l , then the spectrum can change from singular continuous to pure point. The related problem of whether local perturbations to the potential can change the nature of the spectrum in a similar way, is the subject of the present work. We shall see that Example 3 of [5] is a special case of our results when the local perturbation is identically zero.

As far as Lemma 2, this work follows more less the same approach as [3] where it was proved that local perturbations can change singular continuous spectrum to absolutely continuous one. Beginning with Lemma 3 we move along a different sort of ideas which were in part inspired by [2]. The paper proceeds as follows. In Section 2 we use Gel'fand-Levitan's theorem to construct a differential operator L generated by l with singular continuous spectrum, define the perturbed operator \tilde{L} and state the main result. In Section 3 we prove some lemmas which will help us to prove that some set does not contain enough of its limit points and it is therefore countable. In Section 4 we prove that this set is a support of the measure generated by the spectral function of the perturbed operator and from here it follows that \tilde{L} has pure point spectrum.

Manuscript received 9 August 1989.

American Journal of Mathematics 113 (1990), 203–217.

Abusing of the notation, we shall denote with the same symbol the Lebesgue-Stieltjes measure generated by a monotone function and the function itself.

2. Statement of the main result. We shall use the symbols \mathbf{C} , \mathbf{R} , \mathbf{R}^+ to denote the set of complex, real and nonnegative real numbers respectively. We utilize the usual notation for open intervals (a, b) , closed intervals $[a, b]$ and half open intervals $(a, b]$ or $[a, b)$, with end points a, b .

Let $\rho : \mathbf{R} \rightarrow \mathbf{R}$ be defined as follows

$$\rho(\lambda) = \begin{cases} 0 & \text{if } \lambda \in (-\infty, 0] \\ \nu(\lambda) & \text{if } \lambda \in (0, 1] \\ 1 - \frac{2}{\pi} + \frac{2}{\pi} \sqrt{\lambda} & \text{if } \lambda \in (1, \infty) \end{cases}$$

where $\nu(\lambda)$ is the Cantor ternary function (see [14]). The function ρ is continuous and satisfies the conditions of the theorem of Gel'fand-Levitan (see [11]). Therefore there exists a potential

$$q : \mathbf{R}^+ \rightarrow \mathbf{R}$$

and $\alpha \in [0, 2\pi)$ such that the operator L generated by the differential expression

$$(lu)(x) = -u''(x) + q(x)u(x) \quad 0 \leq x < \infty$$

and the boundary condition

$$u(0)\cos \alpha + u'(0)\sin \alpha = 0$$

has ρ for its spectral function.

Let $v : \mathbf{R}^+ \rightarrow \mathbf{R}$ be a continuous function with compact support $S \subset \mathbf{R}^+$.

Let us define the selfadjoint operator \tilde{L} as the one generated by the differential expression:

$$\tilde{lu} = -u'' + \{q(x) + v(x)\}u \quad x \in [0, \infty)$$

and the boundary condition

$$u(0) \cos \beta + u'(0) \sin \beta = 0 \quad \beta \in [0, 2\pi).$$

Let $I := [0, 1]$. Our main result is the following:

THEOREM. *If $\alpha \neq \beta$ or $v(\cdot) \not\equiv 0$ a.e, then the operator \tilde{L} has only pure point spectrum in I .*

Since the operator L has by construction singular continuous spectrum in I , the theorem asserts that if we change the boundary condition or if we perturb the potential locally then the singular continuous spectrum disappears and we have only pure point spectrum. In the case $v \equiv 0$ this result has been known for comparatively long time. See [5] and a related result in [1].

3. Some lemmas. Before we prove the theorem we need some lemmas. The reals α and β are defined as in the previous section.

Consider a fundamental system $\{u_1(x, z), u_2(x, z)\}$ of solutions of

$$lu_k = -u_k''(x) + q(x)u_k(x) = zu_k(x) \quad k = 1, 2 \quad 0 \leq x < \infty$$

which satisfy the conditions

$$u_1(0, z) \cos \alpha + u_1'(0, z) \sin \alpha = 0$$

$$u_2(p, z) = \sin \gamma$$

$$u_2'(p, z) = -\cos \gamma$$

where $p \in \mathbf{R}$ is such that $S \subset [0, p)$, that is to say, p is to the right of the support of $v(x)$ and γ is a point of the open interval $(0, \pi)$.

Consider also a fundamental system $\{\tilde{u}_1(x, z), \tilde{u}_2(x, z)\}$ of solutions of

$$\tilde{l}\tilde{u}_k = -\tilde{u}_k''(x) + \{q(x) + v(x)\}\tilde{u}_k = z\tilde{u}_k(x)$$

$$k = 1, 2 \quad 0 \leq x < \infty$$

such that \tilde{u}_1 and \tilde{u}_2 satisfy the conditions

$$\tilde{u}_1(0, z) \cos \beta + \tilde{u}'_1(0, z) \sin \beta = 0$$

$$\tilde{u}_2(p, z) = \sin \gamma$$

$$\tilde{u}'_2(p, z) = -\cos \gamma.$$

It is known (see [6]) that if z is nonreal, there is a function $m(z)$ such that

$$m(z)u_1(x, z) + u_2(x, z) \in L_2(0, \infty).$$

We call this function the Weyl-Titchmarsh-Kodaira coefficient (W.T.K. henceforth) of L with respect to $\{u_1(x, z), u_2(x, z)\}$.

Let \tilde{m} be the W.T.K. coefficient of \tilde{L} with respect to $\{\tilde{u}_1(x, z), \tilde{u}_2(x, z)\}$.

LEMMA 1. *Let $\lambda \in \mathbf{C}$ be such that $\text{Im } \lambda > 0$.*

Then we have

$$\tilde{m}(\lambda) = \frac{m(\lambda)}{C_1(\lambda) - C_2(\lambda)m(\lambda)}$$

where C_1 and C_2 are the following analytic functions

$$C_1(\lambda) = \frac{W(\tilde{u}_1, u_2)}{W(u_1, u_2)}(\lambda)$$

$$C_2(\lambda) = \frac{W(\tilde{u}_1, u_1)}{W(u_2, u_1)}(\lambda).$$

For the proof see [3].

Let p and γ be chosen as above.

We define the operator L_α as the operator generated through the differential expression

$$lu = -u'' + q(x)u \quad x \in [0, p]$$

and the boundary conditions

$$u(0)\cos \alpha + u'(0)\sin \alpha = 0$$

$$u(p)\cos \gamma + u'(p)\sin \gamma = 0 \quad \gamma \in (0, \pi).$$

Similarly we define the operator L_β as the operator generated by the differential expression

$$\tilde{L}u = -u'' + \{q(x) + v(x)\}u \quad x \in [0, p]$$

and the boundary conditions

$$u(0)\cos \beta + u'(0)\sin \beta = 0$$

$$u(p)\cos \gamma + u'(p)\sin \gamma = 0 \quad \gamma \in (0, \pi).$$

L_α and L_β are operators generated by differential expressions which are regular in $[0, p]$ and therefore their spectra consist only of isolated eigenvalues.

LEMMA 2. *If L_α and L_β do not have exactly the same spectrum we have*

$$W(\tilde{u}_1, u_1)(x, \lambda) \neq 0.$$

The proof is similar to that of Lemma 2 of [3].

Let us define the following sets

$$S_m = \{\xi \in \mathbf{R} \mid \lim_{\zeta \rightarrow \xi} |m(\zeta)| = \infty\}$$

$$S_{\tilde{m}} = \{\xi \in \mathbf{R} \mid \lim_{\zeta \rightarrow \xi} |\tilde{m}(\zeta)| = \infty\}$$

$$A = \{\lambda_i, \mu_i, \gamma_i \in [0, 1] \mid W(u_1, u_2)(\lambda_i) = 0,$$

$$W(\tilde{u}_1, u_1)(\mu_i) = 0, W(\tilde{u}_1, \tilde{u}_2)(\gamma_i) = 0\}.$$

The analytic functions $W(u_1, u_2)(\cdot)$ and $W(\tilde{u}_1, \tilde{u}_2)(\cdot)$ cannot vanish identically since otherwise the selfadjoint operators L_α and L_β would have nonreal eigenvalues. If we assume that L_α and L_β do not have

exactly the same spectrum, then in virtue of Lemma 2 it follows that $W(\tilde{u}_1, u_1)(\cdot)$ do not vanish identically either. From now on we assume this is the case. At the end of Section 4 we shall give conditions which guarantee that both spectra are not identical.

LEMMA 3.

$$[S_m \setminus A] \cap S_{\bar{m}} = \phi.$$

Proof. From Lemma 1 it follows that

$$(1) \quad \left| \frac{C_1(\lambda)}{C_2(\lambda)} - m(\lambda) \right| \left| \tilde{m}(\lambda) + \frac{1}{C_2(\lambda)} \right| = \left| \frac{C_1(\lambda)}{C_2(\lambda)} \right|$$

for every $\lambda \in \mathbf{C}$ with $\text{Im } \lambda > 0$.

If $\xi \in S_m \setminus A$ we have $\lim_{\lambda \rightarrow \xi} |m(\lambda)| = \infty$ and $C_2(\xi) \neq 0$.

This implies

$$\lim_{\lambda \rightarrow \xi} \left| \frac{C_1(\lambda)}{C_2(\lambda)} - m(\lambda) \right| = \infty.$$

Since $\lim_{\lambda \rightarrow \xi} |C_1(\lambda)/C_2(\lambda)|$ exists and is finite it follows from (1) that

$$\lim_{\lambda \rightarrow \xi} \left| \tilde{m}(\lambda) + \frac{1}{C_2(\lambda)} \right| = 0.$$

Therefore $\xi \notin S_{\bar{m}}$ and we have proved

$$S_{\bar{m}} \cap [S_m \setminus A] = \phi. \quad \text{Q.E.D.}$$

From this it follows immediately that

$$[S_{\bar{m}} \setminus A] \cap S_m = [S_{\bar{m}} \setminus A] \cap [S_m \setminus A] = \phi.$$

Let

$$D_{\xi} \rho = \lim_{\epsilon \downarrow 0} \frac{\rho(\xi + \epsilon) - \rho(\xi - \epsilon)}{2\epsilon}$$

be the symmetric derivative of ρ at the point ξ .

Let us define the following sets

$$S'_m = \{\xi \in I \mid D_\xi \rho = \infty\}$$

$$S'_{\tilde{m}} = \{\xi \in I \mid D_\xi \tilde{\rho} = \infty\}.$$

Here $\tilde{\rho}$ denotes the spectral function of \tilde{L} .

LEMMA 4.

$$[S'_m \setminus A] \subset S_m \quad \text{and} \quad [S'_{\tilde{m}} \setminus A] \subset S_{\tilde{m}}.$$

Proof. We know that (see [4])

$$-\operatorname{Im} \frac{\tilde{m}(u + i\epsilon)}{W(\tilde{u}_1, \tilde{u}_2)(u + i\epsilon)} - \operatorname{Im} H(u + i\epsilon) = \int_{-\infty}^{\infty} \frac{\epsilon}{(u - \mu)^2 + \epsilon^2} d\tilde{\rho}(\mu)$$

for $u \in I \setminus A$, where H is an analytic function which is real when the argument is real.

From this it follows as in Lemma 7 of [3] that

$$\left| \frac{\tilde{m}(u + i\epsilon)}{W(\tilde{u}_1, \tilde{u}_2)(u + i\epsilon)} \right| - \operatorname{Im} H(u + i\epsilon) \geq \frac{\tilde{\rho}(u + \epsilon) - \tilde{\rho}(u - \epsilon)}{2\epsilon}.$$

Hence

$$[S'_{\tilde{m}} \setminus A] \subset S_{\tilde{m}}.$$

The other contention can be proved analogously. Q.E.D.

From Lemma 4 it follows that

$$[S'_m \setminus A] \subset [S_m \setminus A]$$

and

$$[S'_{\tilde{m}} \setminus A] \subset [S_{\tilde{m}} \setminus A].$$

From Lemma 3 we have

$$[S'_m \setminus A] \cap [S_m \setminus A] = \phi.$$

Therefore it follows that

$$[S'_m \setminus A] \cap [S'_m \setminus A] = \phi$$

and

$$(2) \quad S'_m \cap [S'_m \setminus A] = \phi.$$

Let $\sigma(T)$ denote the spectrum of the operator T .

LEMMA 5.

$$S'_m \subset \sigma(\tilde{L}) \quad \text{and} \quad S'_m \subset \sigma(L).$$

Proof. We shall prove that

$$\xi \in S'_m \Rightarrow \tilde{\rho}(\xi + \epsilon) - \tilde{\rho}(\xi - \epsilon) > 0$$

for every $\epsilon > 0$.

If $\xi \in S'_m$ then

$$\lim_{\epsilon \downarrow 0} \frac{\tilde{\rho}(\xi + \epsilon) - \tilde{\rho}(\xi - \epsilon)}{2\epsilon} = \infty.$$

Hence given $M > 0$ there exists $k > 0$ such that $0 < \epsilon < k$ implies

$$\frac{\tilde{\rho}(\xi + \epsilon) - \tilde{\rho}(\xi - \epsilon)}{2\epsilon} > M$$

and therefore

$$\tilde{\rho}(\xi + \epsilon) - \tilde{\rho}(\xi - \epsilon) > 2\epsilon M > 0$$

for every $\epsilon < k$.

Since $\tilde{\rho}$ is a nondecreasing function the same follows for $\epsilon \geq k$.

We have proved that $\bar{\rho}$ is strictly increasing in ξ . This implies that $\xi \in \sigma(\tilde{L})$, see [6]. The other contention follows analogously. Q.E.D.

The essential spectrum of an operator T , denoted by $\sigma_{ess}(T)$ is the set of accumulation points of $\sigma(T)$. The absolutely continuous spectrum of T , denoted by $\sigma_{ac}(T)$ is defined as the spectrum of T restricted to its subspace of absolute continuity (see [10]).

We have the following result

LEMMA 6.

$$\sigma_{ess}(L) = \sigma_{ess}(\tilde{L}) \quad \text{and} \quad \sigma_{ac}(L) = \sigma_{ac}(\tilde{L}).$$

Proof. Let A_1 be the selfadjoint realization of l in $[0, p]$ and let \tilde{A}_1 be a selfadjoint realization of \tilde{l} in $[0, p]$. Let A_2 be a selfadjoint realization of l in $[p, \infty)$. Since the differential expressions l and \tilde{l} are the same for $x \geq p$, A_2 is also a selfadjoint realization of \tilde{l} in $[p, \infty)$.

Let us consider the operators $A_1 \oplus A_2$ and $\tilde{A}_1 \oplus A_2$ defined in the space $L_2(0, p) \oplus L_2(p, \infty)$. We have

$$\sigma_{ess}(A_1 \oplus A_2) = \sigma_{ess}(A_1) \cup \sigma_{ess}(A_2)$$

and

$$\sigma_{ac}(A_1 \oplus A_2) = \sigma_{ac}(A_1) \cup \sigma_{ac}(A_2).$$

Similar equalities hold with \tilde{A}_1 instead of A_1 .

Since A_1 and \tilde{A}_1 have only discrete spectrum we have

$$\sigma_{ess}(A_1 \oplus A_2) = \sigma_{ess}(A_2) = \sigma_{ess}(\tilde{A}_1 \oplus A_2)$$

and the same for the absolutely continuous spectrum.

The resolvents of L and $A_1 \oplus A_2$ differ by an at most 2-dimensional operator and the same happens with the resolvents of \tilde{L} and $\tilde{A}_1 \oplus A_2$. Therefore from Weyl's essential spectrum theorem (see [12], p. 112), it follows that $\sigma_{ess}(L) = \sigma_{ess}(A_1 \oplus A_2)$ and $\sigma_{ess}(\tilde{L}) = \sigma_{ess}(\tilde{A}_1 \oplus A_2)$.

From the results of scattering theory with trace class methods (see [13]) the same follows for the absolutely continuous spectrum.

Therefore we have

$$\sigma_{ess}(L) = \sigma_{ess}(A_1 \oplus A_2) = \sigma_{ess}(A_2) = \sigma_{ess}(\tilde{A}_1 \oplus A_2) = \sigma_{ess}(\tilde{L})$$

and the analogous result for σ_{ac} .

Q.E.D.

Let C denote the Cantor ternary set.

LEMMA 7.

$$S'_m = C = \sigma_{ess}(L) \cap I.$$

Proof. Let $x \in C$ and let $\{\epsilon_m\}_{m=1}^\infty$ be an arbitrary sequence converging to zero. For each ϵ_m there exists natural numbers M and n such that

$$\frac{1}{3^M} > \epsilon_m \geq \frac{1}{3^M + 1} \geq \frac{1}{3^n}$$

holds. It can then be proven that

$$\frac{\nu(x + \epsilon_m) - \nu(x - \epsilon_m)}{2\epsilon_m} \geq \frac{1}{2^{n+1}\epsilon_m} > \frac{3^M}{2^{n+1}}.$$

When $\epsilon_m \rightarrow 0$ we must have $n \rightarrow \infty$ and from this it follows that the symmetric derivative of the Cantor function at the point x equals infinity. Therefore we have

$$C \subset S'_m.$$

Now assume that $x \in I \setminus C$. Then for some $\epsilon > 0$ we have

$$\nu(x + \epsilon) - \nu(x - \epsilon) = 0$$

and therefore $x \notin \sigma(L)$, and from Lemma 5 it follows that $x \notin C$. Therefore we have proved $S'_m = C$.

The other equality follows using Lemma 5 and the fact that C is a perfect set.

Q.E.D.

4. Proof of the theorem. We shall prove first that the set S'_m is countable. Let E be the set of limit points of S'_m . From Lemma 5 we know that $S'_m \subset \sigma(\tilde{L})$. Since the spectrum is a closed set we have

$$E \subset \sigma_{\text{ess}}(\tilde{L})$$

and using Lemma 6 follows

$$E \subset \sigma_{\text{ess}}(L) \cap I.$$

Hence from Lemma 7 we obtain

$$E \subset S'_m = C.$$

From here it follows that

$$E \cap S'_m \subset S'_m \cap S'_m.$$

Now

$$S'_m \cap S'_m = S'_m \cap [(S'_m \setminus A) \cup A] = [S'_m \cap (S'_m \setminus A)] \cup [S'_m \cap A].$$

From (2) we have

$$S'_m \cap S'_m = \phi \cup [S'_m \cap A] = S'_m \cap A.$$

Therefore we obtain

$$E \cap S'_m \subset S'_m \cap A.$$

Since the points in A are roots of analytic functions it follows that $E \cap S'_m$ is at most countable.

Since an uncountable set contains always uncountable many of its limit points, (see [15], p. 45) we have to conclude that S'_m is at most countable.

Now let T denote the operator L or \tilde{L} and let τ be the corresponding spectral function. As before let $I := [0, 1]$. We say that a measure m is purely singular if it has a support of Lebesgue measure zero. A support is a set M such that $m(X) = m(X \cap M)$ for every measurable set X .

LEMMA 8. $I \cap \sigma_{ac}(T) = \phi \Leftrightarrow \tau$ is purely singular in I .

Proof. \Rightarrow) given $f \in L_2(0, \infty)$ we can write

$$f = f_s + f_{ac}$$

where $f_s \in H_s(T)$ and $f_{ac} \in H_{ac}(T)$. $H_{ac}(H_s)$ denotes the subspace of absolute continuity (of singularity) with respect to T (see [10]).

For every Borel set $\Delta \subset I$ it follows that

$$\begin{aligned} m_f(\Delta) &:= \langle E(\Delta)f, f \rangle = \langle E(\Delta)(f_s + f_{ac}), f_s + f_{ac} \rangle \\ &= \langle E(\Delta)f_s, f_s \rangle + \langle E(\Delta)f_{ac}, f_s \rangle + \langle E(\Delta)f_s, f_{ac} \rangle + \langle E(\Delta)f_{ac}, f_{ac} \rangle \end{aligned}$$

where $E(\lambda)$ is the spectral family of T .

Since $H_{ac} \perp H_s$ we have

$$\langle E(\Delta)f, f \rangle = \langle E(\Delta)f_s, f_s \rangle + \langle E(\Delta)f_{ac}, f_{ac} \rangle.$$

The assumption $I \cap \sigma_{ac}(T) = \phi$ implies $\langle E(\Delta)f_{ac}, f_{ac} \rangle = 0$, (see [11]). Therefore the measure

$$m_f(\cdot) := \langle E(\cdot)f, f \rangle$$

is purely singular.

Since for every Borel set $\Delta \subset I$ we can choose $f, g \in L_2(0, \infty)$ such that

$$\tau(\Delta) = \langle E(\Delta)f, g \rangle \leq m_f(\Delta)^{1/2} \|g\|$$

holds, it follows that τ is purely singular too.

\Leftarrow) It is easy to see that for every $f \in L_2(0, \infty)$ the measure m_f is absolutely continuous with respect to τ . Since τ is purely singular this implies that $m_f(\cdot)$ is purely singular for every $f \in L_2(0, \infty)$. Therefore for every $f \in H_{ac}$ we have that $m_f = 0$. This implies

$$I \cap \sigma_{ac}(T) = \phi. \qquad \text{Q.E.D.}$$

Proof of the theorem. Since $\rho(\lambda)$ is the Cantor ternary function for $\lambda \in I$ we have that ρ is purely singular and therefore using Lemma

8 it follows that $I \cap \sigma_{ac}(L) = \phi$. Now from Lemma 6 we obtain that $I \cap \sigma_{ac}(\tilde{L}) = \phi$ and from Lemma 8 again, it follows that $\tilde{\rho}$ is purely singular.

Now denote by $\overline{D}_\xi \tilde{\rho}$ the upper and by $\underline{D}_\xi \tilde{\rho}$ the lower symmetric derivative of $\tilde{\rho}$ at a point ξ , these being defined respectively as the upper and as the lower limit of the ratio

$$\frac{\rho(\xi + \epsilon) - \rho(\xi - \epsilon)}{2\epsilon}$$

where ϵ tends to zero.

Let A_∞ denote the set of points $\xi \in I$ at which one at least of the derivatives $\overline{D}_\xi \tilde{\rho}$ and $\underline{D}_\xi \tilde{\rho}$ is infinite, then for any bounded set X which is measurable we have

$$\tilde{\rho}(X) = \tilde{\rho}(X \cap A_\infty) + \int_X \tilde{\rho}'(x) dx.$$

For the proof of this see [16, p. 151].

Now since $\tilde{\rho}$ is singular we have

$$\tilde{\rho}(X) = \tilde{\rho}(X \cap A_\infty).$$

Clearly $S'_m \subset A_\infty$. Let $J := A_\infty \setminus S'_m$. It is easy to see that $\tilde{\rho}$ is continuous for every point in J . Therefore from [15, p. 125] it follows that

$$\tilde{\rho}(J) = 0.$$

Then we have

$$\begin{aligned} \tilde{\rho}(X) &= \tilde{\rho}(X \cap [S'_m \cup J]) \\ &= \tilde{\rho}(X \cap S'_m) + \tilde{\rho}(X \cap J) \\ &= \tilde{\rho}(X \cap S'_m). \end{aligned}$$

Therefore S'_m is a support of $\tilde{\rho}$.

Since S'_m is at most countable it follows that $\tilde{\rho}$ is discrete. Hence

the spectrum of \tilde{L} in I contains a set of eigenvalues which is dense in the Cantor set C .

Up to this point we have assumed $\sigma(L_\alpha) \neq \sigma(L_\beta)$. Now to finish the proof of the theorem we shall give conditions which imply this assumption.

In Hald [7] the following result is proven:

Consider the eigenvalue problems

$$(3) \quad \begin{aligned} -u'' + q(x)u &= \lambda u \\ hu(0) - u'(0) &= 0, \quad Hu(\pi) + u'(\pi) = 0 \end{aligned}$$

$$(4) \quad \begin{aligned} -u'' + \tilde{q}(x)u &= \tilde{\lambda}u \\ \tilde{h}u(0) - u'(0) &= 0, \quad \tilde{H}u(\pi) + u'(\pi) = 0 \end{aligned}$$

where q and \tilde{q} are integrable on $[0, \pi]$. Let λ_j and $\tilde{\lambda}_j$ be the eigenvalues of equations (3) and (4) and assume that $\lambda_j = \tilde{\lambda}_j$ for all j . If $q(x) = \tilde{q}(x)$ for almost all x in the interval $\pi/2 \leq x \leq \pi$ and if $H = \tilde{H}$, then $q(x) = \tilde{q}(x)$ almost everywhere and $h = \tilde{h}$.

Now choose $p \in \mathbf{R}$ such that $S \subset (0, (1/2)p)$ where S is the support of the perturbation v . By scaling we can take in the above result $[0, p]$ instead of $[0, \pi]$ and $\tilde{q}(x) = q(x)$ on $[(1/2)p, p]$ instead of $\tilde{q}(x) = q(x)$ on $[\pi/2, \pi]$. Assuming $\alpha \neq 0$, $\beta \neq 0$ we can therefore conclude the following: If $\alpha \neq \beta$ or if $v(x) \neq 0$ a.e. then L_α and L_β do not have the same spectrum.

When $\alpha = \beta = 0$ the same follows using a result of Hochstadt-Lieberman [8].

In the case $\alpha = 0$, $\beta \neq 0$ or $\alpha \neq 0$, $\beta = 0$ it can be seen from the asymptotic behavior of the eigenvalues (see for example Remark 1 of [9]) that $\sigma(L_\alpha) \neq \sigma(L_\beta)$.

Thus the theorem is completely proven.

Q.E.D.

Remark. If in [3] we define L_α and L_β as it was done in this work, the results of [3] can be generalized in the obvious way.

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